'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 6th Semester Examination, 2023

DSE-P3-MATHEMATICS

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.

The question paper contains DSE-3A and DSE-3B. Candidates are required to answer any *one* from the *two* courses and they should mention it clearly on the Answer Book.

DSE-3A

POINT SET TOPOLOGY

GROUP-A

Answer any *four* questions $3\times4 = 12$

1. Let X denotes two element set $\{0, 1\}$. Prove that X^w is uncountable. 3

2. Let $\{\tau_\alpha\}$ be a family of topologies on a non-empty set X. Show that $\bigcap_\alpha \tau_\alpha$ is also a topology on X. 3

3. Prove or disprove — the union of an infinite number of closed sets in a topological space is closed. 3

4. Show that $\mathbb{Z}_+$ is not finite. 3

5. Prove that $|[0, 1]| = c$. 3

6. Show that the product of two Hausdorff spaces is Hausdorff. 3

GROUP-B

Answer any *four* questions $6\times4 = 24$

7. Let β be a family of subsets of X satisfying the conditions: 6

   (i) $\forall x \in X,\ \exists\, B \in \beta$ such that $x \in B$.

   (ii) $\forall B_1, B_2 \in \beta$ with $x \in B_1 \cap B_2$, $\exists\ B_3 \in \beta$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

   Let τ be the collection of all possible union of members of β. Prove that τ is a topology on X.

8. For any set X, prove that $|P(X)| = 2^{|X|}$. 6

9. (a) Show that every infinite set has an enumerable subset. 3

(b) Let A be an enumerable set and $a \in A$ be fixed. If $A' = A \setminus \{a\}$, show that A and A' are equipotent. 3

10. Show that a topological space X is Hausdorff iff the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$. 6

11. Let $p : X \to Y$ be a closed, continuous and surjective map such that for every point $y \in Y$, $p^{-1}(\{y\})$ is compact in X. Show that if Y is compact, so is X. 6

12. State and prove Cantor's Theorem. 6

GROUP-C

Answer any *two* questions $12 \times 2 = 24$

13.(a) Show that if A, B, C are pairwise disjoint sets with $|A| = \alpha$, $|B| = \beta$, $|C| = \gamma$, then 4

$$(\alpha^{\beta})^{\gamma} = \alpha^{\beta\gamma}.$$

(b) If n is any natural number and A be any infinite set, then prove that $n + |A| = |A|$. 4

(c) For $a, b \in \mathbb{R}$, define $a < b$ if $b - a$ is positive and rational. Show that '$<$' is a strict partial order on $\mathbb{R}$. What are the maximal simply ordered subsets of $\mathbb{R}$? 4

14.(a) Let A and B be proper subsets of the connected spaces X and Y respectively. Show that $(X \times Y) \setminus (A \times B)$ is connected. Hence conclude that $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is connected. 6

(b) Define limit-point compactness. Show that compact space is limit-point compact but not conversely. 6

15.(a) State and prove Baire Category Theorem. 6

(b) Let $f : X \to Y$ be a bijective mapping. Prove that the following are equivalent: 6

(i) f is a homeomorphism.

(ii) f is open and continuous.

(iii) f is closed and continuous.

(iv) $f(\overline{A}) = \overline{f(A)}$ for any $A \subseteq X$.

16.(a) Show that the topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$. 6

(b) Investigate the convergence and the possible limit(s) of the sequence $\{x_n = \frac{1}{n}\}$ in the cofinite topology on $\mathbb{R}$. 6

DSE-3B

BOOLEAN ALGEBRA AND AUTOMATA THEORY

GROUP-A

Answer any *four* questions from the following 3×4 = 12

1. Let L be a lattice and $a, b \in L$. Then show that $a \le b \Rightarrow a \vee c \le b \vee c$ and $a \wedge c \le b \wedge c$ for any $c \in L$.

2. For any two elements x, y in a lattice L, let $[x, y] = \{a \in L : x \le a \le y\}$. Show that $[x, y]$ is a sub-lattice of L.

3. Consider the poset $P = (P(\mathbb{N}), \subseteq)$. Define a map $\phi : P \to P$ by

$$\phi(\upsilon) = \begin{cases} \{1\} & \text{if} \quad 1 \in \upsilon \\ \{2\} & \text{if} \quad 2 \in \upsilon \text{ and } 1 \notin \upsilon \\ \Phi & \text{otherwise} \end{cases}$$

Is ϕ an order isomorphism? Justify your answer.

4. Construct a finite automata equivalent to the regular expression:

(i) $L = ab(aa+bb)(a+b)^* b$

(ii) $L = aa^*(a+b)^*$

5. Find the parse tree for generating the string 0100110 from the following grammar:

$S \to 0S/1AA$

$A \to 0/1A/0B$

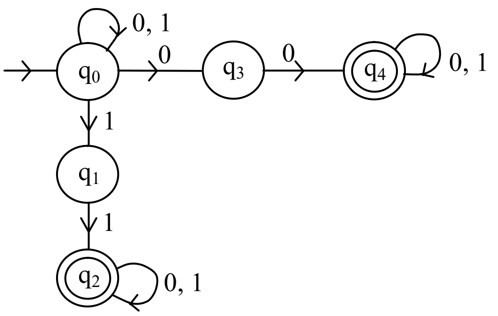
$B \to 1/0BB$

6. Construct a logic circuit that produces $(x+y+z)(xyz)'$ as its output.

GROUP-B

Answer any *four* questions from the following 6×4 = 24

7. (a) Convert the NFA to equivalent DFA: 3



(b) Prove that every regular language is a context-free language. 3

8. (a) Check whether the following grammar is ambiguous or not: 3

$S \rightarrow 0Y/01$

$X \rightarrow 0XY/0$

$Y \rightarrow XY1/1$

(b) Design a DFA that accepts the following language 3

$L = \{x \in \{0, 1\}^* : x$ is of even length and begins with $01\}$.

9. (a) Determine the DNF of the following Boolean expression: 3

$(x+y+z)(xy+x'z')'$

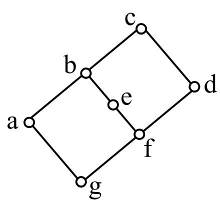
(b) Draw the switching circuits which realizes the following Boolean expressions: 3

(i) $x(yz+y'z')+x'(yz'+y'z)$

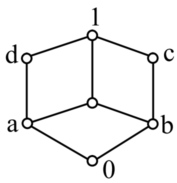
(ii) $(x+y+z+u)(x+y+u)(x+z)$

10.(a) Determine which of the following lattices are modular or distributive: 4

(i)



(ii)



(b) Find an NFA M on $\Sigma = \{a, b\}$ such that $L(M) = \{ab, ba\}$. 2

11.(a) Construct a PDA to accept the following language: 4

$L = \{a^n b^{2n} : n \geq 1\}$.

(b) For any Boolean algebra B, prove that $(a+b)(b+c)(c+a) = ab+ba+ca$ for all $a, b, c \in B$. 2

12.(a) Let L be a complemented distributive lattice. Prove that $(a \vee b)' = a' \wedge b'$ and $(a \wedge b)' = a' \vee b'$ for all $a, b \in L$. 3

(b) Show that in a complemented distributive lattice, the following are equivalent 3

(i) $a \leq b$ (ii) $a \wedge b' = 0$ (iii) $a' \vee b = 1$ (iv) $b' \leq a'$

GROUP-C

Answer any *two* questions from the following 12×2 = 24

13.(a) Design a Turing machine that accepts the language $L = \{O^{2^n} : n \geq 0\}$. 6

(b) State and prove pumping lemma for regular languages and hence show that the language $\{a^p : p$ is a prime integer$\}$ is not regular. 4+2

14.(a) Consider the following ε-NFA. 3+3

	ε	a	b	c
$\rightarrow p$	ϕ	$\{p\}$	$\{q\}$	$\{r\}$
q	$\{p\}$	$\{q\}$	$\{r\}$	ϕ
$*r$	$\{q\}$	$\{r\}$	ϕ	$\{p\}$

(i) Compute the ε-closure of each state.

(ii) Convert the automaton to an equivalent DFA.

(b) Transform the following DNF to a CNF: 3

$x_1'x_2'x_3' + x_1'x_2'x_3 + x_1x_2'x_3' + x_1x_2x_3'$

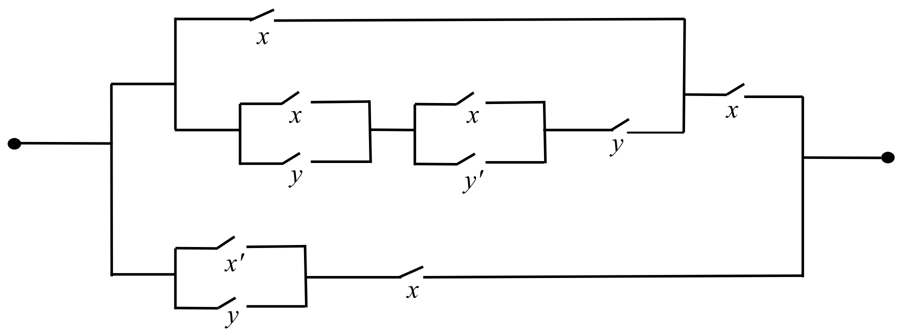
(c) Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined by 3

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ x & \text{if } x \text{ is odd} \end{cases}$$

Prove that f is order preserving.

15.(a) Suppose a 3-variable Boolean Term is given as follows: $\phi = xy + xz' + yz$. Minimize ϕ using K-map. 4

(b) Obtain a Boolean expression which represents the following circuit. Moreover; draw an equivalent circuit as simple as you can: 4



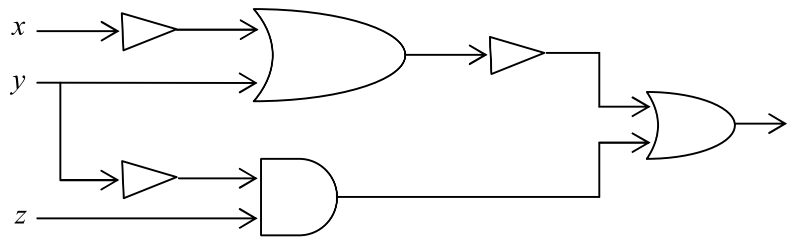
(c) Suppose 4

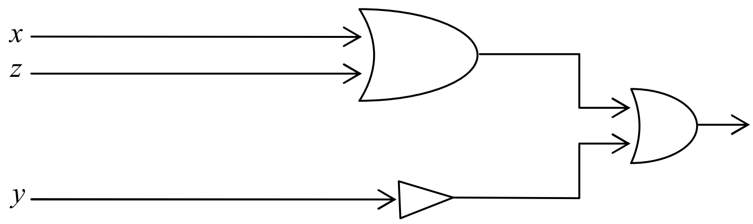
$\phi(a, b, c, d) = \Sigma m(0, 1, 3, 7, 8, 9, 11, 15)$.

Minimize ϕ using Quine-McCluskey method.

16.(a) A room has three entrances and at each entrance there is a switch to independently control the light in the room in such a way that flicking any one of the switches changes the state of the light (on to off and off to on). Design a switching circuit to accomplish this. 6

(b) Show that the following logic circuits given by the figures below are equivalent. 6





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